MATHEMATICS

1. If $f(x)=\begin{cases}x-2 & 0\le x\le 2\\ -2 & -2\le x\le 0\end{cases}$ and h(x) = f(|x|) + |f(x)| then $\int_0^k h(x)\,dx$ is equal to (k > 0)

(1) 0 (2) $\frac{k}{2}$ (3) 2k (4) k

Ans. (1)

Sol. $f(|x|)=\begin{cases}-2-x, & x<0\\ x-2, & x>0\end{cases}$ $|f(x)|=\begin{cases}2, x<0\\ 2-x, & x>0\end{cases}$

$\Rightarrow h(x)=f(|x|)+|f(x)|=\begin{cases}-x, & x<0\\ 0, & x>0\end{cases}$

$\Rightarrow \int_0^k h(x)dx=\int_0^k 0dx=0$

2. There are three bags A, B and C. Bag A contain 7 Black balls and 5 Red balls, Bag B contains 5 Red and 7 Black balls and Bag C contain 7 Red and 7 Black balls. A ball is drawn and found to be black find probability that it is drawn from Bag A.

Ans. $(\frac{7}{18})$

Sol.

$$Prob=\frac{\frac{7}{12}}{\frac{7}{12}+\frac{5}{12}+\frac{7}{14}}$$

$$=\frac{\frac{7}{6}}{\frac{7}{6}+\frac{5}{6}+1}$$

$$=\frac{7}{7+5+6}=\frac{7}{18}$$

3. Find the number of rational numbers in the expansion of $\left(2^{\frac{1}{5}}+5^{\frac{1}{3}}\right)^{15}$.

Ans. (2)

Sol. $T_{r+1}={}^{15}C_r\left(2^{1/5}\right)^{15-r}\left(5^{1/3}\right)^r$

$={}^{15}C_r 2^{3-\frac{r}{5}}.5^{\frac{r}{3}}$; r = 3K & 5K

There r = 0; 15

So Total No. of Rational Terms are "2".

4. Find value of $\int_{0}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\sin x \cos x} dx$

Ans. $\left(\frac{\pi}{3\sqrt{3}}\right)$

Sol. $\Rightarrow I = \int_{0}^{\pi/2} \frac{\cos^2 x dx}{1+\sin x \cos x}$

$\therefore \; 2I = \int_{0}^{\pi/2} \frac{2dx}{2+\sin 2x}$

$I = \int_{0}^{\pi/2} \frac{dx}{2+\frac{2\tan x}{1+\tan^2 x}}$

$2I = \int_{0}^{\pi/2} \frac{\sec^2 x dx}{\tan^2 x + \tan x + 1}$

$2I = \int_{0}^{\infty} \frac{dt}{t^2 + t + 1}$

$2I = \int_{0}^{\infty} \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$2I = \frac{1}{\sqrt{3}/2}\left[\tan^{-1}\left(\frac{t+\frac{1}{2}}{\sqrt{3}/2}\right)\right]_{0}^{\infty}$

$I = \frac{1}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{6}\right]$

$I = \frac{\pi}{3\sqrt{3}}$

5. If $x^2 - ax + b = 0$ has roots 2, 6; and $\alpha = \frac{1}{2a+1}$; $\beta = \frac{1}{2b-a}$. Find equation having roots α, β.

Ans. $(272x^2 - 33x + 1 = 0)$

Sol. $a = 2 + 6 = 8$

$b = 2 \times 6 = 12$

$\alpha = \frac{1}{17}; \beta = \frac{1}{16}$

Required $EQ^n = x^2 - \left(\frac{1}{17} + \frac{1}{16}\right)x + \frac{1}{17} \times \frac{1}{16}$

$\Rightarrow 272x^2 - 33x + 1 = 0$

6. $\lim_{x \to 4} \frac{(5+x)^{\frac{1}{3}} - (1+2x)^{\frac{1}{3}}}{(5+x)^{\frac{1}{2}} - (1+2x)^{\frac{1}{2}}}$

Ans. $\left(\frac{2 \times 9^{1/3}}{9}\right)$

Sol. $\lim_{x\to 4}\frac{(5+x)^{1/3}-(1+2x)^{1/3}}{(5+x)^{1/2}-(1+2x)^{1/2}}$

$\frac{(9+h)^{1/3}-(9+2h)^{1/3}}{(9+h)^{1/2}-(9+2h)^{1/2}}=\frac{9^{1/3}\left[\frac{h}{27}-\frac{2h}{27}\right]}{3\left(\frac{h}{18}-\frac{h}{9}\right)}$

$=\frac{9^{1/3}}{3}\frac{\left(\frac{-h}{27}\right)}{\frac{-h}{18}}$

$=\frac{2\times 9^{1/3}}{9}$

7. AB, BC, CA are sides of triangle having 5, 6, 7 points respectively. How many triangles are possible using these points.

Ans. (751)

Sol. ${}^{18}C_3-{}^{5}C_3-{}^{6}C_3-{}^{7}C_3$

= 17 × 16 × 3 – 10 – 20 – 35

= 816 – 65 = 751

8. 2, p and q are in G.P. in an A.P. 2 is third term, p is 7^{th} term and q is 8^{th} term find p and q.

Ans. $(P=\frac{1}{2},q=\frac{1}{8})$

Sol. p = 2r, q = $2r^2$

In A.P. A + 2d = 2

A + 6d = 2r

A + 7d = $2r^2$

By Solving $r=\frac{1}{4}$

$P=\frac{1}{2},q=\frac{1}{8}$

9. If the domain of the function $\sin^{-1}\left(\frac{3x-22}{2x-19}\right)+\log_e\left(\frac{3x^2-8x+5}{x^2-3x-10}\right)$ is [α, β] then 3α + 10β is equal to

(1) 100 (2) 95 (3) 97 (4) 98

Ans. (3)

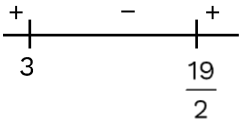
Sol. $-1\le\frac{3x-22}{2x-19}\le 1$

$\frac{3x-22}{2x-19}+1\ge 0$

$\frac{5x-41}{2x-19}\ge 0\Rightarrow x\in\left(-\infty,\frac{41}{5}\right]\cup\left(\frac{19}{2},\infty\right)$

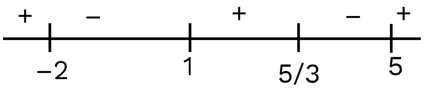
$$\frac{3x-22}{2x-19}-1\le 0$$

$$\frac{x-3}{2x-19}\le 0\Rightarrow x\in\left[3,\frac{19}{2}\right)$$



$$\frac{3x^2-3x-5x+5}{x^2-5x+2x-10}>0$$

$$\frac{(3x-5)(x-1)}{(x-5)(x+2)}>0$$



$$\Rightarrow\left[5,\frac{41}{5}\right]$$

$$=3\times 5+10\times\frac{41}{5}$$

= 15 + 82 = 97

10. x + (2sin2θ) y + 2cos2θ = 0

x + (sinθ) y + cosθ = 0

x + (cosθ) y − sinθ = 0

find nontrivial solution

Ans. $\left(\alpha=\cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)\right)$

Sol.
$$\begin{vmatrix}1 & 2\sin 2\theta & 2\cos\theta\\ 1 & \sin\theta & \cos\theta\\ 1 & \cos\theta & -\sin\theta\end{vmatrix}=0$$

$$1\left[-\sin^2\theta-\cos^2\theta\right]-2\sin 2\theta[-\sin\theta-\cos\theta]+2\cos 2\theta[\cos\theta-\sin\theta]=0$$

$$-1+2\sin 2\theta(\sin\theta+\cos\theta]+2\cos 2\theta[\cos\theta-\sin\theta]=0$$

$$-1+2\sin\theta\sin 2\theta+2\sin 2\theta\cos\theta+2\cos\theta\cos 2\theta-2\cos 2\theta\sin\theta=0$$

$$-1+2\cos\theta+2\sin\theta=0$$

$$\sin\theta+\cos\theta=\frac{1}{2}$$

$$\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=\frac{1}{2\sqrt{2}}$$

$$\cos\left(\theta-\frac{\pi}{4}\right)=\cos\alpha$$

$$\theta-\frac{\pi}{4}=2n\pi\pm\alpha$$

where $\alpha=\cos^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

11. Let $f(x) = x^5 + 2e^{x/4}$ for all $x \in R$. consider a function $(gof)(x) = x$ for all $x \in R$. Then the value of $8g'(2)$ is

(1) 4 (2) 16 (3) 8 (4) 2

Ans. (2)

Sol. $g(f(x)) = x$

$g'(f(x)).f'(x) = 1$

$g'(f(x)) = \frac{1}{f'(x)}$; $f'(x) = 5x^4 + \frac{1}{2}e^{x/4}$

$g'(2) = \frac{1}{f'(0)} = \frac{1}{2/4} = 2$ $f'(0) = \frac{1}{2}$

$8'g'(2) = 16$

12. Let $f(x) = \frac{2x^2 - 3x + 9}{2x^2 + 3x + 4}$. If maximum value of f(x) is m and minimum value of f(x) is n then find

m + n?

Ans. (10)

Sol. $y = \frac{2x^2 - 3x + 9}{2x^2 + 3x + 4}$

$y(2x^2 + 3x + 4) = 2x^2 - 3x + 9$

$(y - 1)2x^2 + 3x(y + 1) + 4y - 9 = 0$

If $y \neq 1 \Rightarrow D \geq 0$

$9(y + 1)^2 - 4(y - 1)(4y - 9) \geq 0$

$9(y^2 + 2y + 1) - 4(4y^2 - 9y - 4y + 9) \geq 0$

$9y^2 - 16y^2 + 18y + 52y + 9 - 36 \geq 0$

$-7y^2 + 70y - 27 \geq 0$

$7y^2 - 70y + 27 \leq 0$ has roots α and β $y = \frac{70 \pm \sqrt{4900 - 4 \times 7 \times 27}}{2 \times 7}$

$\Rightarrow \alpha \leq y \leq \beta$ $y = \frac{70 \pm \sqrt{4144}}{14}$

$\alpha = m = \frac{70 - \sqrt{4144}}{14}$

$\beta = n = \frac{70 + \sqrt{4144}}{14}$

$= m + n = 10$

13. $f(x) = \begin{cases} \frac{1 - \cos 2x}{x^2} & x < 0 \\ \alpha & x = 0 \\ \beta\frac{\sqrt{1 - \cos x}}{x} & x > 0 \end{cases}$. If f(x) is continuous at x = 0 find $\alpha^2 + \beta^2$.

Ans. (12)

Sol. $\lim_{x\to 0^-}\frac{1-\cos 2x}{x^2} = 2 = \alpha = \lim_{x\to 0^+}\beta\sqrt{\frac{1-\cos x}{x^2}} = \frac{\beta}{\sqrt{2}}$.

Hence $\alpha^2 + \beta^2 = 4 + 8 = 12$

14. Let α and β be the sum and the product of all the nonzero solutions of the equation $(\bar{z})^2 + |z| = 0,\ z \in C$ then $4(\alpha^2 + \beta^2)$ is equal to

(1) 6 (2) 2 (3) 4 (4) 8

Ans. (4)

Sol. $\bar{z}^2 + |z| = 0$

$x^2 - y^2 - 2xyi + \sqrt{x^2+y^2} = 0$

$x = 0$

$y^2 = \sqrt{y^2}$

$y^2 = |y|$ $y = 1, -1$

$i, -i$

$y = 0$

$x^2 + \sqrt{x^2+y^2} = 0$ No non zero solution

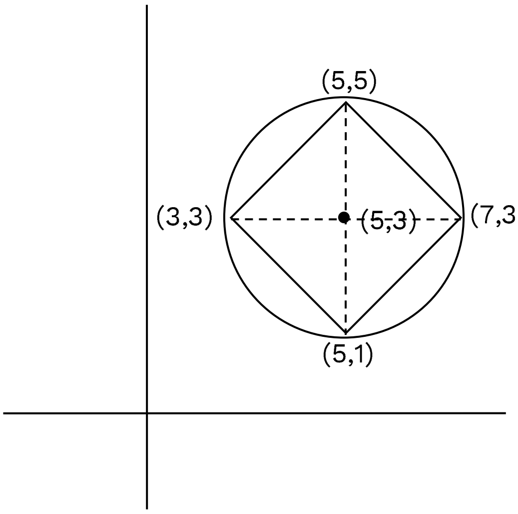
$\alpha = 0$ $\beta = 1$

$4(\alpha^2 + \beta^2) = 4$

15. A square is inscribed in the circle $x^2 + y^2 - 10x - 6y + 30 = 0$. One side of this square is parallel to $y = x + 3$. If (x_i, y_i) are the vertices of the square, then $\sum(x_i^2 + y_i^2)$ is equal to:

(1) 148 (2) 156 (3) 152 (4) 160

Ans. (3)



Sol.

$\sum x_i^2 + y_i^2 = 25 + 25 + 49 + 9 + 25 + 1 + 9 + 9 = 152$

16. If differential equation satisfies $\frac{dy}{dx} - y = \cos x$ at $x = 0$, $y = \frac{-1}{2}$. Find $y\left(\frac{\pi}{4}\right)$.

Ans. (0)

Sol. $\frac{dy}{dx} - y = \cos x$

$I \cdot f = e^{\int -1dx} = e^{-x}$

$y \cdot e^{-x} = \int e^{-x} \cdot \cos x dx$

$I = \int e^{-x} \cos x dx$

$I = \left(-e^{-x}\right)\cos x - \int(-\sin x)\left(-e^{-x}\right)dx$

$I = -e^{-x}\cos x - \int e^{-x}\sin x dx$

$I = -e^{-x}\cos x - \left[\left(-e^{-x}\right)\sin x + \int e^{-x}\cos x dx\right]$

$I = -e^{-x}\cos x + e^{-x}\sin x - I$

$2I = e^{-x}(\sin x - \cos x)$

$y \cdot e^{-x} = \frac{e^{-x}(\sin x - \cos x)}{2} + c$

$y = \frac{(\sin x - \cos x)}{2} + c$

c = 0

$y\left(\frac{\pi}{4}\right) = \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{2} = 0$

17. Let α, β, $\in$ R. Let the mean and the variance of 6 observations –3, 4, 7, - 6, α, β be 2 and 23 respectively. The mean deviation about the mean of these 6 observations is

(1) $\frac{11}{3}$ (2) $\frac{16}{3}$ (3) $\frac{13}{3}$ (4) $\frac{14}{3}$

Ans. (3)

Sol. $\bar{x} = 2 = \frac{-3+4+7-6+\alpha+\beta}{6} \Rightarrow \alpha + \beta = 10$

$\sigma^2 = 23 = \frac{(-3-2)^2 + (4-2)^2 + (7-2)^2 + (-6-2)^2 + (\alpha-2)^2 + (\beta-2)^2}{6}$

$\Rightarrow \alpha^2 + \beta^2 = 52$

$\therefore$ $\alpha = 6$ & $\beta = 4$

$\therefore$ M. D. about mean = $\frac{13}{3}$

18. $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{k}$, $\vec{c}$ is an unit vector making angle 60° with $\vec{a}$ and 45° with $\vec{b}$. Find $\vec{c}$

Ans. (1)

Sol. Let $\vec{c} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k}$, where $2C_1 + 2C_2 - C_3 = \frac{3}{2}$

$C_1 - C_2 = 1$

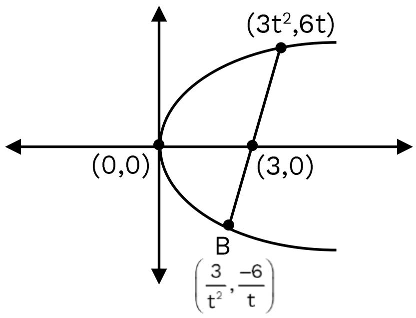
$C_1^2 + C_2^2 + C_3^2 = 1.$

19. If the length of focal chord of $y^2 = 12x$ is 15 and if the distance of the focal chord from origin is p then $10p^2$ is equal to

(1) 36 (2) 25 (3) 72 (4) 144

Ans. (3)

Sol.



$y^2 = 4(3)x$; a = 3 $\Rightarrow$ focus = (3,0)

$t_1t_2 = -1$

$A = 3t^2, 6t$

then $B = \frac{3}{t^2}, \frac{-6}{t}$

AB = length of focal chord

$= a(t_1 - t_2)^2$

$= 3\left(t + \frac{1}{t}\right)^2 = 15$

$3\left(t + \frac{1}{t}\right)^2 = 15$

$t + \frac{1}{t} = \sqrt{5}$

$t - \frac{1}{t} = \sqrt{\left(t + \frac{1}{t}\right)^2 - 4}$

$t - \frac{1}{t} = 1$

$m_{AB} = \frac{6t - \frac{6}{t}}{3t^2 - \frac{3}{t^2}}$

$m_{AB} = \frac{2}{t - \frac{1}{t}}$

$\therefore m_{AB} = 2$

Equation of AB: $y - 0 = 2(x-3)$

$y = 2x-6$

$2x-y-6 = 0$

Distance from origin, $P = \frac{|2(0) - 0 - 6|}{\sqrt{2^2 + 1}} = \frac{6}{\sqrt{5}}$

$10P^2 = \frac{10 \times 36}{5} = 72$

20. Shortest distance between lines $\frac{x+1}{-2}=\frac{y}{2}=\frac{z-1}{1}$ and $\frac{x-5}{2}=\frac{y-2}{-3}=\frac{z-1}{1}$ is $\frac{38k}{6\sqrt{5}}$, find

$\int_0^k [x^2]\,dx$

Ans. $(5-\sqrt{2}-\sqrt{3})$

Sol. $S.D=\frac{(6\hat{i}+2\hat{j}).(5\hat{i}+4\hat{j}+2\hat{k})}{\sqrt{45}}=\frac{38}{3\sqrt{5}}=\frac{38k}{6\sqrt{5}}\Rightarrow k=2$

$$\int_0^2 [x^2]dx=\int_1^{\sqrt{2}} dx+\int_{\sqrt{2}}^{\sqrt{3}} 2dx+\int_{\sqrt{3}}^{2} 3dx=(\sqrt{2}-1)+2(\sqrt{3}-\sqrt{2})+3(2-\sqrt{3})=5-\sqrt{2}-\sqrt{3}$$

21. y = y(x) is a solution of the differential equation

$\left(x^4+2x^3+3x^2+2x+2\right)dy-\left(2x^2+2x+3\right)dx=0$. If $y(0)=\frac{\pi}{4}$. Find y(–1)

Ans. $(-\frac{\pi}{4})$

Sol. $\frac{dy}{dx}=\frac{2x^2+2x+3}{x^4+2x^3+3x^2+2x+2}$

$\frac{dy}{dx}=\frac{\left(x^2+1\right)+\left(x^2+2x+2\right)}{\left(x^2+1\right)\left(x^2+2x+2\right.}=\frac{1}{(x+1)^2+1}+\frac{1}{x^2+1}$

Hence y = $\tan^{-1}x + \tan^{-1}(x+1) + c$

If $y(0)=\frac{\pi}{4}\Rightarrow c=0$

So $y(-1)=-\frac{\pi}{4}$

22. Curve y = 1 + 3x – 2x² and y = $\frac{1}{x}$ intersects at point $\left(\frac{1}{2},2\right)$ then area enclosed between curve is $\frac{1}{24}\left(\ell\sqrt{5}+m\right)-n\log_e\left(1+\sqrt{5}\right)$ then find the value of ℓ + m + n is

Ans. (30)

Sol. $1+3x-2x^2=\frac{1}{x}$

$\Rightarrow x+3x^2-2x^3=1$

$\Rightarrow 2x^3-3x^2-x+1=0$

$\Rightarrow 2x^3-x^2-2x^2+x-2x+1=0$

$\Rightarrow (2x-1)x^2-x(2x-1)-1(2x-1)=0$

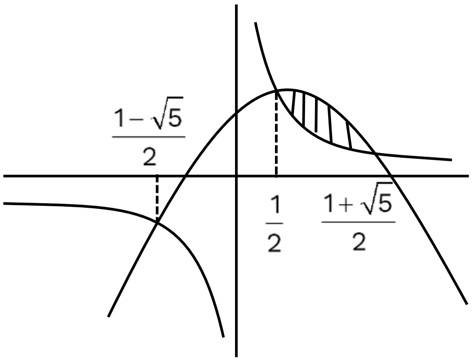
$\Rightarrow (2x-1)\left(x^2-x-1\right)=0$

$x=\frac{1}{2}$ or $x^2-x-1=0$

$x=\frac{1\pm\sqrt{5}}{2}$

$x = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

$$\text{Area} = \int_{1/2}^{\frac{\sqrt{5}+1}{2}} \left(\left(-2x^2+3x+1\right)-\frac{1}{x}\right)dx$$



$$= \left[-\frac{2x^3}{3}+\frac{3x^2}{2}+x-\ln x\right]_{1/2}^{\frac{\sqrt{5}+1}{2}}$$

$$= \left(-\frac{2}{3}\left(\frac{\sqrt{5}+1}{2}\right)^3+\frac{3}{2}\left(\frac{\sqrt{5}+1}{2}\right)^2+\left(\frac{\sqrt{5}+1}{2}\right)-\ln\left(\frac{\sqrt{5}+1}{2}\right)\right)$$

$$-\left(-\frac{1}{12}+\frac{3}{8}+\frac{1}{2}-\ln\frac{1}{2}\right)$$

$$= -\frac{1}{12}(5\sqrt{5}+1+3\sqrt{5}(\sqrt{5}+1))+\frac{3}{8}(6+2\sqrt{5})+\frac{\sqrt{5}+1}{2}$$

$$-\ln(\sqrt{5}+1)+\ln 2-\left(\frac{-2+9+12}{24}\right)-\ln 2$$

$$= -\frac{1}{12}(16+8\sqrt{5})+\frac{3}{4}(3+\sqrt{5})+\frac{\sqrt{5}+1}{2}-\frac{19}{24}-\ln(\sqrt{5}+1)$$

$$= \frac{1}{24}[-32-16\sqrt{5}+54+18\sqrt{5}+12\sqrt{5}+12-19]-\ln(\sqrt{5}+1)$$

$$= \frac{1}{24}[15+14\sqrt{5}]-\ln(\sqrt{5}+1)$$

So $\ell = 14$, m = 15, n = 1

Hence ℓ + m + n = 14 + 15 + 1 = 30